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The affine primitive permutation groups of degree less than 1000

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Abstract

In this paper we complete the classification of the primitive permutation groups of degree less than 1000 by determining the irreducible subgroups of $\mathrm{GL}(n, p)$ for p prime and $p^n < 1000$. We also enumerate the maximal subgroups of $\mathrm{GL}(8, 2)$, $\mathrm{GL}(4, 5)$ and $\mathrm{GL}(6, 3)$.

MSC: 20B10; 20B15; 20H30

1. Some background

The classification of primitive groups of small degree has a long and rich history. We give a brief review of the most significant developments here, and refer the reader to Short's 1991 publication (Short, 1991) for a fuller account. The first major work in this area was a paper published by Jordan (1871) which counts the primitive groups of degrees 4–17. With the exception of one missing group in degrees 9, 12 and 15, eight missing groups of degree 16 and two missing groups of degree 17, this list is complete. In 1874, Jordan correctly stated that every transitive group of degree 19 is either alternating, symmetric or of affine type. Cole in 1893 completely determined the transitive groups of degree 9 (Cole, 1893a,b). In a series of papers at the turn of the century (Miller, 1895, 1896, 1897a,b, 1898a,b, 1900), Miller corrected the work of Jordan by determining the primitive groups of degrees 12–17. Miller also correctly tabulated the number of soluble primitive groups of degree less than 24 (Miller, 1898b). Martin extended the classification by enumerating the primitive groups of degree 18 (Martin, 1901), and Bennett completed the classification of the primitive groups of degree $d \leq 20$ by enumerating the primitive groups of degree 20 (Bennett, 1912).

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In the 1960s, Sims redetermined this list for degree $d \leq 20$ (Sims, 1970). In part, this was achieved by the development of new computational techniques, and by 1970 Sims had extended these methods to classify the primitive groups of degree $d \leq 50$. This latter classification was widely circulated, despite never being published, and was made available in 1977 as the PRMGPS database in V3.5 of CAYLEY (Cannon, 1984). It was later turned into databases in both GAP (GAP Group, 2002) and MAGMA (Bosma et al., 1997; Bosma and Cannon, 2002, pp. 634–638). Buekenhout and Leemans (1996) determined the abstract structure of the groups in these databases. In the early 1980s, Pogorolev independently determined the primitive groups with insoluble socles of degrees 21–64 (Pogorelov, 1980a,b, 1982). This classification was done by hand, and only listed the groups up to abstract isomorphism, rather than up to permutation isomorphism.

Dramatic progress was made in the late 1980s. Il'in and Takmakov (1986) published a list of the simple primitive groups of degree less than 1000, which they had calculated by hand. Then Dixon and Mortimer (1988) used the Classification of Finite Simple Groups and the O'Nan–Scott theorem to classify the primitive groups with insoluble socles for all degrees less than 1000. They pointed out some errors in the work of Il'in and Takmakov as they did so. This list was implemented into GAP by Theißen, along with the primitive affine groups of degree less than 256 (Theißen, 1997). However, several groups were missing from both Dixon and Mortimer's and Theißen's lists.

Thus since 1987 the major open problem has been to classify the primitive groups with soluble socles of degree less than 1000. To do this, one needs to determine the primitive affine subgroups of $\mathrm{AGL}(n, p)$ for prime p and $p^n < 1000$. There is a homomorphism ψ from the stabilizer of the identity in $\mathrm{AGL}(n, p)$, denoted $\mathrm{AGL}(n, p)_0$, to $\mathrm{GL}(n, p)$, and a group $G \leq \mathrm{AGL}(n, p)$ is primitive if and only if $\psi(G_0)$ is irreducible. This problem is therefore equivalent to that of determining the irreducible subgroups of $\mathrm{GL}(n, p)$, up to conjugacy in $\mathrm{GL}(n, p)$, for $p^n < 1000$.

Many people have worked on classifications of linear groups; we list only those papers which are relevant to our purposes. Harada and Yamaki in 1979 listed the irreducible subgroups of $\mathrm{GL}(n, 2)$ for $n \leq 6$. The irreducible subgroups of $\mathrm{GL}(7, 2)$ were listed by Kondrat'ev (1985). By 1987 he had calculated the irreducible insoluble subgroups of $\mathrm{GL}(8, 2)$ and $\mathrm{GL}(9, 2)$ (Kondrat'ev, 1987, 1986). These calculations were performed by hand, and representations were not given. By the early 1990s, increases in both the processing power and storage capabilities of desktop computers provided the incentive to extend these classifications to cover a wider range of groups. In 1991 Short used computational techniques to list the soluble subgroups of $\mathrm{GL}(n, p)$ for $p^n < 256$ (Short, 1991), and gave representations for those which he found (some subgroups of $\mathrm{GL}(2, 11)$ and $\mathrm{GL}(2, 13)$ were missing). These were made available as both GAP and MAGMA (Bosma and Cannon, 2002, pp. 642–645) databases of linear groups, and were incorporated into Theißen's primitive group database in GAP. After completing all cases in this paper except for $\mathrm{GL}(6, 3)$, we heard of Eick and Höfling's recent classification of the irreducible soluble subgroups of $\mathrm{GL}(n, p)$ for $p^n < 6561$ (Eick and Höfling, in press).

The purpose of this paper is to complete the classification of the primitive groups of degree less than 1000 by computing the irreducible subgroups of $\mathrm{GL}(n, p)$ for $p^n < 1000$. In particular we classify the irreducible subgroups of $\mathrm{GL}(4, 5)$ and of $\mathrm{GL}(6, 3)$, and recompute the irreducible subgroups of $\mathrm{GL}(8, 2)$ and $\mathrm{GL}(9, 2)$. Our technique is highly automated, and makes heavy use of the new algorithms in Cannon and Holt (in press) for computing maximal subgroups. We have found all conjugacy classes of irreducible subgroups of $\mathrm{GL}(n, p)$ for prime p and $p^n < 1000$, and have re-examined Dixon and Mortimer's 1988 classification of nonaffine primitive groups, correcting some minor errors. We have also constructed a representation of each primitive group, and in the affine case we have computed representations of the corresponding irreducible linear groups. MAGMA V2.10 includes a database of all of the primitive groups of degree less than 1000, and a function to convert the primitive groups of affine type into the corresponding irreducible matrix groups. The irreducible matrix groups may also be downloaded from http://magma.maths.usyd.edu.au/users/colva.

In Section 2 we present some mathematical preliminaries, and describe Aschbacher's classification of subgroups of linear groups. We then define two lists $\mathcal{M}_1$ and $\mathcal{M}_2$ of subgroups of $\mathrm{GL}(n, p)$, before presenting our principal algorithm, `IrreducibleSubgroups`$(\mathcal{M}_1, \mathcal{M}_2)$, which computes all irreducible subgroups of $\mathrm{GL}(n, p)$. In Sections 3 and 4 we analyse $\mathrm{GL}(4, 5)$ and $\mathrm{GL}(6, 3)$ respectively, and in Section 5 we briefly examine both $\mathrm{GL}(8, 2)$ and $\mathrm{GL}(9, 2)$. Section 6 discusses the primitive groups of degree less than 1000. In Section 7 we describe the ways in which we have checked the accuracy of our classification, and how it depends on the work of others, before providing a summary of our results in Section 8.

2. Introductory material

2.1. Mathematical preliminaries

In this section we recall some basic mathematical definitions and results, before describing how we calculate the input to our subgroup algorithm.

We start with a few elementary permutation group definitions, both Cameron (1999) and Dixon and Mortimer (1996) are useful references. A permutation group G acting on a set Ω is *primitive* if G is transitive and preserves no proper nontrivial equivalence relation on Ω. The *socle* of a finite group is the product of its minimal normal subgroups.

Let $V := \mathbb{F}_p^{(n)}$. An *affine transformation* of V is a map $t_{a,w} : V \to V$ where $a \in \mathrm{GL}(V)$, $w \in V$ and $t_{a,w}(v) := va + w$. The group of all affine transformations forms the *affine general linear group*, denoted $\mathrm{AGL}(n, p)$. We consider $\mathrm{AGL}(n, p)$ as a permutation group, acting on the p^n vectors of V. The socle of $\mathrm{AGL}(n, p)$ may then be identified with $(V, +)$, and the point stabilizer with $\mathrm{GL}(n, p)$. A subgroup $G \leq \mathrm{AGL}(n, p)$ is called a *group of affine type* if $V \trianglelefteq G$. Recall that G is primitive if and only if its point stabilizer is an irreducible subgroup of $\mathrm{GL}(V)$.

There are many versions of the O'Nan–Scott theorem, see Dixon and Mortimer (1996) for a full discussion. The version used by Dixon and Mortimer in their 1988 classification (Dixon and Mortimer, 1988) states that a finite primitive permutation group G must belong to one of five classes. One of these is the groups with soluble socle, these are precisely the groups of affine type.

Next we recall some elementary properties of linear groups. Recall that a subgroup G of $\mathrm{GL}(n, p)$ is *irreducible* if G stabilizes no proper nontrivial subspace of $\mathbb{F}_p^{(n)}$. We say that

G is *absolutely irreducible* if the image of G under the natural embedding into $\mathrm{GL}(n, \mathbb{F})$ is irreducible for all field extensions $\mathbb{F}$ of $\mathbb{F}_p$. We say that a group $G \leq \mathrm{GL}(V)$ is *imprimitive* if G is irreducible and preserves a direct sum decomposition $V = V_1 \oplus \cdots \oplus V_t$. Here t is a divisor of n, and $\dim(V_i) = n/t$ for $1 \leq i \leq t$.

Theorem 2.1 (Aschbacher's theorem (Aschbacher, 1984)). *Let $G \leq \mathrm{GL}(n, q)$ be given, let $q = p^e$, let $V := \mathbb{F}_q^n$ and let $Z := Z(\mathrm{GL}(n, q))$. Then one of the following holds:*

1. *G is reducible.*
2. *G is imprimitive.*
3. *A conjugate of G can be embedded in $\Gamma\mathrm{L}(n/s, q^s)$ for some prime s dividing n.*
4. *G preserves a tensor product decomposition $V = V_1 \otimes V_2$, where $\dim V_1 \neq \dim V_2$.*
5. *A conjugate of G can be embedded in $\mathrm{GL}(n, q_0)Z$ for some subfield $\mathbb{F}_{q_0}$ of $\mathbb{F}_q$, of prime index.*
6. *The dimension $n = r^m$ is a prime power. If r is odd or $n = 2$ then r divides $p-1$ and G normalizes an extraspecial r-group. Otherwise 4 divides $p - 1$ and G normalizes a 2-group of symplectic type.*
7. *G preserves a tensor induced decomposition $V = V_1 \otimes \cdots \otimes V_t$.*
8. *$G \leq N_{\mathrm{GL}(n,q)}(C)$ for some classical group C.*
9. *For some non-abelian simple group T, the group $G/(G \cap Z)$ is almost simple with socle T. In this case the normal subgroup $(G \cap Z) \cdot T$ acts absolutely irreducibly, and preserves no nondegenerate classical form.*

The original theorem describes classes of subgroups of all classical groups, allowing one to "recurse" on many of the classes. The theorem as stated here describes only the linear case: we refer the reader to Aschbacher (1984) and Kleidman and Liebeck (1990) for a full discussion.

Groups lying in class i of this classification are called $\mathcal{C}_i$ *groups*. Groups lying in $\cup_{i=1}^{8} \mathcal{C}_i$ are called *geometric groups*. We will take no further interest in the reducible groups.

Let $i \in \{2, \ldots, 8\}$, and let $G \in \mathcal{C}_i$ be given. Then G is called *potentially maximal* if G is not a proper subgroup of a geometric group. We define all $\mathcal{C}_9$ groups to be *potentially maximal*. The motivation here is to describe all groups that could possibly be maximal as potentially maximal, without examining them too closely. It is clear that any maximal subgroup of $\mathrm{GL}(n, p)$ is a potentially maximal group.

We write $G \cdot H$ to denote an extension of a group G by a group H. For a split extension we write $G : H$, and for a central product we write $G \circ H$.

When naming groups, the natural number n is the cyclic group of order n, and D_{2n} is the dihedral group of order $2n$. The symbol p^{1+2k} denotes an extraspecial p-group.

In general we follow the notation of Kleidman and Liebeck (1990) for the classical groups, however for the sake of clarity we will briefly review some notation. Let $\Gamma\mathrm{L}(n, p^e)$ be the full semilinear group over $\mathbb{F}_{p^e}^{(n)}$, so that $\Gamma\mathrm{L}(n, p^e) = \mathrm{GL}(n, p^e) : e$ with the cyclic group acting as field automorphisms. For odd q, let $\mathrm{SL}^{\pm}(n, q)$ be the subgroup of $\mathrm{GL}(n, q)$ consisting of all matrices of determinant ± 1. Let $\mathrm{O}^{\epsilon}(n, q)$, where ϵ is $+$, $-$ or omitted, be the largest subgroup of $\mathrm{GL}(n, q)$ which preserves a nondegenerate quadratic form of type ϵ. The subgroup of $\mathrm{O}^{\epsilon}(n, q)$ consisting of all matrices of determinant 1 is $\mathrm{SO}^{\epsilon}(n, q)$.

There is a subgroup of $\mathrm{SO}^\epsilon(n, q)$ of index 1 or 2 which is simple modulo scalars. We write $\Omega^\epsilon(n, q)$ for this subgroup.

We will require two lists of subgroups of $G := \mathrm{GL}(n, p)$. The first list, $\mathcal{M}_1$, consists of the potentially maximal subgroups of G. To compute $\mathcal{M}_1$ we first use Aschbacher's theorem to identify each conjugacy class of potentially maximal geometric subgroups of G. We also compute a list $\mathcal{L}$ consisting of all simple groups of order dividing $|G|$. For each $T \in \mathcal{L}$, and for each $A \leq \mathrm{Out}(T)$, we then use ad hoc methods to determine whether any group $Z \cdot T \cdot A$ should be added to $\mathcal{M}_1$. These methods will be described separately below, as the various cases are considered.

The almost simple groups database of MAGMA V2.9 stores the maximal subgroups of all almost simple groups with socle of size less than 1.6×10^7. One may compute the maximal subgroups of a permutation group G whenever each of its non-abelian simple composition factors is described in the database.

The list $\mathcal{M}_2$ contains enough subgroups of the groups in $\mathcal{M}_1$ to ensure that all further subgroups may be computed automatically. That is to say, $\mathcal{M}_2$ contains all irreducible groups H such that $H \leq_{\max} K$ for some $K \in \mathcal{M}_1 \cup \mathcal{M}_2$, but $H \not\leq L$ for any $L \in \mathcal{M}_1 \cup \mathcal{M}_2$ whose maximal subgroups may be computed automatically. Note that we have given only a sufficient condition for a group to be in $\mathcal{M}_2$, not a necessary one. The methods used to determine $\mathcal{M}_2$ will be described on a case-by-case basis.

2.2. The algorithm

In this subsection, we describe our algorithms. The main algorithm for finding the irreducible subgroups of $\mathrm{GL}(n, p)$ is called `IrreducibleSubgroups()`. This takes as input the two lists $\mathcal{M}_1$ and $\mathcal{M}_2$ described in Section 2.1, and returns a list $\mathcal{I}$ of conjugacy class representatives of irreducible subgroups of $\mathrm{GL}(n, p)$.

The function `HasComputableSubgroups`(G) returns true if and only if the maximal subgroups of G may be computed automatically, that is if all non-abelian simple composition factors of G are in the almost simple groups database. A full description of the function `IsGLConjugate`(A, B) will be given in Roney-Dougal (in preparation). In brief, it starts by computing various properties of the structures and actions of A and B to try to prove that they are not conjugate under $\mathrm{GL}(n, p)$. If this fails to distinguish between them then SMASH (Holt et al., 1996) and various other algorithms for identifying the Aschbacher class of a group are used to search efficiently for a conjugating element.

`IrreducibleSubgroups`($\mathcal{M}_1$, $\mathcal{M}_2$)

1. Set $\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2$.
2. Set $\mathcal{I} := \emptyset$.
3. For $G \in \mathcal{M}$ do
   (a) Add G to $\mathcal{I}$.
   (b) If `HasComputableSubgroups`(G) then
      - Let $\mathcal{S}$ be the set of irreducible maximal subgroups of G, up to conjugacy in G.
      - For each $S \in \mathcal{S}$, if there does not exist a group $T \in \mathcal{M}$ such that `IsGLConjugate`(S, T) returns true, add S to $\mathcal{M}$.

4. Return the list $\mathcal{I}$.

Several methods were used to compute the maximal irreducible subgroups of $\mathrm{GL}(n, p)$. Wherever possible we used theoretical methods to classify them, and checked our answers using a variant of the main algorithm called `GLMaximals()`. We describe `GLMaximals()` here, and defer the description of the theoretical methods until we need them. The set $\mathcal{N}$ will be defined in more detail later on, for now it suffices to state that $\mathcal{N} \subseteq \mathcal{M}_1$.

`GLMaximals`$(\mathcal{M}_1, \mathcal{M}_2, \mathcal{N})$

1. Set $\mathcal{M} := \mathcal{M}_1 \cup \mathcal{M}_2$.
2. Set $a := \mathrm{Min}\{|G| : G \in \mathcal{N}\}$.
3. For $G \in \mathcal{M}$ do
   (a) If `HasComputableSubgroups`(G) then
      - Let $\mathcal{S}$ be the set of irreducible maximal subgroups of G, up to conjugacy in G.
      - Let $\mathcal{T} := \{S : S \in \mathcal{S}, |S| \geq a\}$.
      - For $S \in \mathcal{T}$, if there does not exist a group $T \in \mathcal{M}$ such that `IsGLConjugate`(S, T) returns true, append S to $\mathcal{M}$. Otherwise, delete from $\mathcal{N}$ all groups that are conjugate to S.
4. Return the list $\mathcal{N}$.

3. The irreducible subgroups of GL(4, 5)

In this section we describe our classification of the irreducible subgroups of $\mathrm{GL}(4, 5)$. The first stage in this classification is to apply Aschbacher's theorem to find the potentially maximal geometric subgroups of $\mathrm{GL}(4, 5)$.

Throughout this section, let $Z := Z(\mathrm{GL}(4, 5))$.

Lemma 3.1. *The irreducible maximal geometric subgroups of* $\mathrm{GL}(4, 5)$ *lie in the following list.*

- $\mathcal{C}_2$. *Imprimitive groups:* $\mathrm{GL}(1, 5) \wr \mathrm{Sym}(4)$, $\mathrm{GL}(2, 5) \wr \mathrm{Sym}(2)$.
- $\mathcal{C}_3$. *Superfield groups:* $\Gamma\mathrm{L}(2, 25)$.
- $\mathcal{C}_6$. *Symplectic* 2*-group normalizers:* $(4 \circ 2^{(1+4)}) \cdot \mathrm{Sp}(4, 2)$.
- $\mathcal{C}_7$. *Tensor induced groups:* $(\mathrm{GL}(2, 5) \circ \mathrm{GL}(2, 5)) : \mathrm{Sym}(2)$.
- $\mathcal{C}_8$. *Normalizers of classical groups:* $N_{\mathrm{GL}(4,5)}(\mathrm{O}^-(4, 5))$, $N_{\mathrm{GL}(4,5)}(\mathrm{Sp}(4, 5))$, *and* $\mathrm{SL}^{\pm}(4, 5)$.

Proof. We start with a straightforward application of Aschbacher's theorem, and then refine this list. In the imprimitive case we find one group for each proper divisor of 4, but in the superfield case we need only consider prime divisors. We do not need to consider tensor product groups, as the only proper factorization of 4 is into two equal numbers. Since 5 is prime we do not need to consider subfield groups. The extraspecial-type group is in fact a 2-group of symplectic type, since 4 is a proper power of 2, and 5 is a prime that is congruent to 1 mod 4. One may use the algorithms described in Holt et al. (1996) and

Leedham-Green and O'Brien (2002) to verify that the group $N_{\mathrm{GL}(4,5)}(\mathrm{O}^+(4,5))$ is tensor induced. □

The next step is to examine the list of simple groups of order dividing $|\mathrm{GL}(4,5)|$, to see if there are any potentially maximal $\mathcal{C}_9$ subgroups of $\mathrm{GL}(4,5)$.

Lemma 3.2. *There are no potentially maximal* $\mathcal{C}_9$ *subgroups of* $\mathrm{GL}(4,5)$.

Proof. Let $\mathcal{L}$ denote the set of all simple groups of order dividing $|\mathrm{GL}(4,5)|$. Each of these other than $\mathrm{PSL}(4,5)$ is described in the Modular Atlas (Jansen et al., 1995).

$$\mathcal{L} = \{\mathrm{Alt}(5), \mathrm{Alt}(6), \mathrm{PSL}(2,25), \mathrm{PSL}(2,31), \mathrm{PSU}(3,4), \mathrm{PSL}(3,5), \mathrm{PSp}(4,5), \mathrm{PSL}(4,5)\}.$$

The Modular Atlas states that for all $G \in \mathcal{L}$ other than $\mathrm{PSL}(4,5)$, and for all H such that $Z \cdot G \leq H \leq Z \cdot \mathrm{Aut}(G)$, all representations of H as an absolutely irreducible subgroup of $\mathrm{GL}(4,5)$ preserve a nondegenerate form when restricted to $Z \cdot G$, and hence are $\mathcal{C}_8$ groups. □

Theorem 3.3. *The group* $\mathrm{GL}(4,5)$ *has* 647 *conjugacy classes of irreducible subgroups, of which* 509 *are soluble and* 138 *are insoluble. The maximal irreducible subgroups of* $\mathrm{GL}(4,5)$ *are the imprimitive and superfield groups listed in Lemma* 3.1 *together with* $\mathrm{SL}^{\pm}(4,5)$.

Proof. It follows from Lemmas 3.1 and 3.2 that we may take $\mathcal{M}_1$ to be

$$\mathcal{M}_1 = \{\mathrm{GL}(1,5) \wr \mathrm{Sym}(4), \mathrm{GL}(2,5) \wr \mathrm{Sym}(2), \Gamma\mathrm{L}(2,25), (4 \circ 2^{1+4}) \cdot \mathrm{Sp}(4,2), (\mathrm{GL}(2,5) \circ \mathrm{GL}(2,5)) : \mathrm{Sym}(2), N_{\mathrm{GL}(4,5)}(\mathrm{O}^-(4,5)), N_{\mathrm{GL}(4,5)}(\mathrm{Sp}(4,5)), \mathrm{SL}^{\pm}(4,5)\}.$$

The only group $G \in \mathcal{M}_1$ for which `HasComputableSubgroups(`G`)` returns false is $\mathrm{SL}^{\pm}(4,5)$. Hence $\mathcal{M}_2 = \{\mathrm{SL}(4,5)\}$, since it follows from Aschbacher's theorem that all other maximal subgroups of $\mathrm{SL}^{\pm}(4,5)$ and $\mathrm{SL}(4,5)$ are subgroups of other groups in $\mathcal{C}_1$.

For the second statement, we examine $\mathcal{M}_1$. The tensor induced groups, the groups normalizing symplectic 2-groups, the group $N_{\mathrm{GL}(4,5)}(\mathrm{Sp}(4,5))$, and $N_{\mathrm{GL}(4,5)}(\mathrm{GO}^{\pm}(4,5))$ can be represented by matrices of determinant ± 1, and hence are subgroups of $\mathrm{SL}^{\pm}(4,5)$. Order and determinant considerations show that $\mathrm{SL}^{\pm}(4,5)$ and $\Gamma\mathrm{L}(2,25)$ are maximal, and that at least one of the imprimitive groups is maximal. Considering the geometry of the situation we see that both imprimitive groups are maximal. □

4. The irreducible subgroups of GL(6, 3)

Throughout this section set $Z := Z(\mathrm{GL}(6,3))$.

Lemma 4.1. *Let G be a maximal irreducible geometric subgroup of* $\mathrm{GL}(6,3)$. *Then G lies in the following list.*

- $\mathcal{C}_2$. *Imprimitive groups:* $\mathrm{GL}(2,3) \wr \mathrm{Sym}(3)$, $\mathrm{GL}(3,3) \wr \mathrm{Sym}(2)$.
- $\mathcal{C}_3$. *Superfield groups:* $\mathrm{GL}(2,27) : 3$, $\mathrm{GL}(3,9) : 2$.
- $\mathcal{C}_4$. *Tensor product groups:* $\mathrm{GL}(2,3) \circ \mathrm{GL}(3,3)$.

- $\mathcal{C}_8$. *Normalizers of classical groups:* $N_{\mathrm{GL}(6,3)}(\mathrm{O}^+(6,3))$, $N_{\mathrm{GL}(6,3)}(\mathrm{O}^-(6,3))$, $N_{\mathrm{GL}(6,3)}(\mathrm{Sp}(6,3))$, *and* $\mathrm{SL}(6,3)$.

Proof. In the imprimitive case, the possible divisors of 6 are 1, 2 and 3. Let $G := \mathrm{GL}(1,3) \wr \mathrm{Sym}(6)$. We construct the natural G-module M, and find a matrix A representing an isomorphism from M to its dual. The matrix A is invertible and symmetric, and therefore represents a nondegenerate symmetric bilinear form preserved by G. We check that each of the generators of G preserves the corresponding quadratic form, and conclude that G is contained in an orthogonal group.

In the superfield case there are two distinct prime divisors of 6, yielding two potentially maximal superfield groups. In the tensor product case there is a unique proper factorization of 6 into two distinct factors.

There are no subfield groups, as 3 is prime. In the extraspecial case we find no groups because 6 is not a prime power, and in the tensor induced case there are no groups since 6 is not a proper power.

Finally, in the classical case we find the full normalizers of each classical subgroup of $\mathrm{GL}(6,3)$. $\square$

We now search for potentially maximal subgroups $G \leq \mathrm{GL}(6,3)$ which lie in Aschbacher class $\mathcal{C}_9$. We call a representation ϕ of a group G *good* if $\phi(G)$ is an absolutely irreducible subgroup of $\mathrm{GL}(6,3)$. Recall that if G is a potentially maximal $\mathcal{C}_9$ group, and the socle of $G/Z(G)$ is isomorphic to T, then the restriction of ϕ to $Z(G) \cdot T$ is also good, and preserves no nondegenerate classical form.

Lemma 4.2. *A potentially maximal* $\mathcal{C}_9$ *subgroup of* $\mathrm{GL}(6,3)$ *is one of the following:* $2 \cdot \mathrm{PSL}(2,11)$, $2 \times \mathrm{PSL}(3,3)$, $2 \cdot M_{12}$.

Proof. Let $\mathcal{L}$ denote the set of all simple groups of order dividing $|\mathrm{GL}(6,3)|$. Those which are not described in the Modular Atlas are marked *.

$$\begin{aligned}\mathcal{L} = \{&\mathrm{Alt}(n) \text{ for } n \in \{5,6,7,8,9\},\\ &\mathrm{PSL}(n,q) \text{ for } (n,q) \in \{(2,7),(2,8),(2,11),(2,13),(2,27),\\ &(2,64)^*,(3,3),(3,4),(3,9),(4,3),(5,3)^*,(6,3)^*\},\\ &\mathrm{PSU}(n,q) \text{ for } (n,q) \in \{(4,2),(4,3),(5,2)\}, \mathrm{PSp}(6,2), \mathrm{PSp}(6,3)^*,\\ &\Omega(7,3)^*, G(2,3), \mathrm{Sz}(8), \text{sporadics } M_{12} \text{ and } M_{22}\}.\end{aligned}$$

For each $G \in \mathcal{L}$ and for each $A \leq \mathrm{Out}(G)$ we must now determine whether $2 \times G \cdot A$ or $2 \cdot G \cdot A$ is a potentially maximal $\mathcal{C}_9$ subgroup of $\mathrm{GL}(6,3)$. Note that it suffices to consider groups with a centre of order 2.

If G is described in the Modular Atlas, and $\mathrm{Out}(G)$ is cyclic, one may immediately determine whether any isoclinic variant of a bicyclic extension of G has a good representation, and if so whether this representation occurs as a subgroup of $\mathrm{Sp}(6,3) : 2$ or $\mathrm{O}^\epsilon(6,3) : 2$, for $\epsilon = \pm 1$.

We analyse those groups with a noncyclic outer automorphism group. In Table 1, we list all pairs $(G, \mathrm{Out}(G))$ where $G \in \mathcal{L}$ and $\mathrm{Out}(G)$ is noncyclic.

Any good representation of $\mathrm{Alt}(6) \cdot 2^2$ must restrict to the unique good representation of $\mathrm{Sym}(6)$. We compute the normalizer in $\mathrm{GL}(6,3)$ of $\mathrm{Sym}(6)$, and find that it preserves

Table 1
Noncyclic automorphism groups

G	$\mathrm{Out}(G)$	G	$\mathrm{Out}(G)$
$\mathrm{Alt}(6)$	2^2	$\mathrm{PSL}(4,3)$	2^2
$\mathrm{PSL}(3,4)$	$2 \times \mathrm{Sym}(3)$	$\mathrm{PSU}(4,3)$	D_8
$\mathrm{PSL}(3,9)$	2^2		

a nondegenerate form. Let $\mathrm{Alt}(6) \cdot 2_i$ be as defined in the $\mathbb{ATLAS}$ (Conway et al., 1985), for $1 \leq i \leq 3$. There is no group isomorphic to $2 \cdot \mathrm{Alt}(6) \cdot 2_3$, and hence none isomorphic to $2 \cdot \mathrm{Alt}(6) \cdot 2^2$. One may use the Modular Atlas to check that all good representations of groups isoclinic to $2 \cdot \mathrm{Alt}(6) \cdot 2_1$ or $2 \cdot \mathrm{Alt}(6) \cdot 2_2$ preserve nondegenerate forms.

We see in Hiss and Malle (2001) that $\mathrm{PSL}(3,4)$ has no good representations. The double cover $2 \cdot \mathrm{PSL}(3,4)$ has a unique good representation, but its normalizer in $\mathrm{GL}(6,3)$ is a $\mathcal{C}_8$ group.

The representations of $\mathrm{PSL}(3,9)$ as a subgroup of $\mathrm{GL}(6,3)$ correspond to $\Gamma\mathrm{L}(3,9)$. We observe that the Schur multiplier of $\mathrm{PSL}(3,9)$ is trivial.

We remark that $\mathrm{PSL}(4,3) \cong \Omega^+(6,3)$, and hence that the good representation of $N_{\mathrm{GL}(6,3)}(\mathrm{O}^+(6,3)) = (2 \times \Omega^+(6,3)) \cdot 2^2$ lies in $\mathcal{C}_8$. The double cover of $\Omega^+(6,3)$ has no good representations.

We remark that $\mathrm{PSU}(4,3) \cong \mathrm{P}\Omega^-(6,3)$. There are no good representations of $\mathrm{P}\Omega^-(6,3)$. The group $N_{\mathrm{GL}(6,3)}(\mathrm{O}^-(6,3)) = 2 \cdot \mathrm{P}\Omega^-(6,3) \cdot D_8$ has a good representation as a $\mathcal{C}_8$ group.

Finally we consider the remaining groups. In Hiss and Malle (2001) we see that $\mathrm{PSL}(2,64)$ has no good representations, and in Lübeck (2001) we see that the same is true for $\mathrm{PSL}(5,3)$. Noting that both $\mathrm{PSL}(2,64)$ and $\mathrm{PSL}(5,3)$ have trivial Schur multiplier completes these cases. Chevalley groups of type $\mathrm{P}\Omega(7,3)$ clearly do not have any good representations. In Lübeck (2001) we see that the only good representation of a Chevalley group of type $\mathrm{PSp}(6,3)$ is the natural representation as $\mathrm{Sp}(6,3)$. □

Theorem 4.3. *The group* $\mathrm{GL}(6,3)$ *has* 471 *conjugacy classes of irreducible subgroups, of which* 324 *are soluble and* 147 *are insoluble. The irreducible maximal subgroups of* $\mathrm{GL}(6,3)$ *are the groups listed in Lemma* 4.1.

Proof. We take $\mathcal{M}_1$ to consist of the groups in Lemmas 4.1 and 4.2.

To construct $\mathcal{M}_2$ we note that there are exactly two groups for which `HasComputableSubgroups()` returns `false`. These are $\Gamma\mathrm{L}(3,9)$ and $\mathrm{Sp}(6,3) : 2$. Let us start by considering the maximal irreducible subgroups of $\Gamma\mathrm{L}(3,9)$. The maximal subgroups of $\mathrm{P}\Gamma\mathrm{L}(3,9)$ are listed in the $\mathbb{ATLAS}$. The imprimitive case includes $8^2 : \mathrm{Sym}(3) \cdot 2$, which we ignore as it will correspond to a subgroup of $\mathrm{GL}(2,3) : \mathrm{Sym}(3)$. The superfield case includes $\mathrm{PSL}(1,3^6)$, which is clearly contained in $\Gamma\mathrm{L}(2,27)$. Thus the only groups which we consider when constructing $\mathcal{M}_2$ are a subfield group $\mathrm{PSL}(3,3) \times 2$, and the classical groups $\mathrm{PSU}(3,3) : 2$ and $\mathrm{Aut}(\mathrm{Alt}(6)) \cong \mathrm{P}\Gamma\mathrm{O}(3,9)$. For each group in turn we take the intersection of the maximal subgroup of $\mathrm{P}\Gamma\mathrm{L}(3,9)$ with $\mathrm{PSL}(3,9)$, and then write down the corresponding subgroup of $\mathrm{GL}(3, 9) = 8 \times \mathrm{PSL}(3, 9)$. This subgroup is then mapped into $\mathrm{GL}(6, 3)$, and its normalizer is computed. Each of these groups is put into $\mathcal{M}_2$, along with all groups G such that $\mathrm{GL}(3, 9) \leq G < \Gamma\mathrm{L}(3, 9)$. Since `HasComputableSubgroups()` returns `true` for each of these three normalizers, we do not need to recurse further at this stage.

Next we consider the maximal irreducible subgroups of $\mathrm{Sp}(6, 3) : 2$. The maximal irreducible subgroups of $\mathrm{PSp}(6, 3)$ are again listed in the $\mathbb{ATLAS}$. There are two imprimitive groups, which are subgroups of $\mathrm{GL}(2, 3) \wr \mathrm{Sym}(3)$ and of $\mathrm{GL}(3, 3) \wr \mathrm{Sym}(2)$. There are two superfield groups which must be subgroups of $\Gamma\mathrm{L}(2, 27)$. There is one $\mathcal{C}_9$ group, $2 \cdot \mathrm{Sym}(5)$, which we add to $\mathcal{M}_2$, along with $\mathrm{Sp}(6, 3)$ itself. Finally we add $2 \cdot \mathrm{PSL}(2, 13)$, the only maximal subgroup of $\mathrm{Sp}(6, 3)$ which does not extend to a subgroup of $\mathrm{Sp}(6, 3) : 2$.

The function `IrreducibleSubgroups(`$\mathcal{M}_1, \mathcal{M}_2$`)` returns the stated number of groups.

To perform the maximal subgroup calculation, we start by noting that $2 \cdot \mathrm{PSL}(2, 11)$, $2 \cdot M_{12}$ and $2 \times \mathrm{PSL}(3, 3)$ are subgroups of $\mathrm{SL}(6, 3)$ and set $\mathcal{N}$ to be $\mathcal{M}_1$ with these groups omitted. The stated results are the output of `GLMaximals(`$\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}$`)`. □

5. The irreducible subgroups of GL(8, 2) and GL(9, 2)

We briefly present our results for $\mathrm{GL}(8, 2)$ and $\mathrm{GL}(9, 2)$.

Lemma 5.1. *Let G be a maximal irreducible subgroup of* $\mathrm{GL}(8, 2)$. *Then G lies in the following list:*

- $\mathcal{C}_2$. *Imprimitive groups:* $\mathrm{GL}(4, 2) \wr \mathrm{Sym}(2)$.
- $\mathcal{C}_3$. *Superfield groups:* $\Gamma\mathrm{L}(4, 4)$.
- $\mathcal{C}_8$. *Normalizers of classical groups:* $\mathrm{Sp}(8, 2)$.

Proof. In the imprimitive case it is clear that $\mathrm{GL}(1, 2) \wr \mathrm{Sym}(8)$ fixes the vector consisting of all 1s, and hence that the group is reducible. We use the techniques described in Lemma 4.1 to show that $\mathrm{GL}(2, 2) \wr \mathrm{Sym}(4)$ preserves a quadratic form. Thus the only potentially maximal subgroup in the imprimitive case is $\mathrm{GL}(4, 2) \wr \mathrm{Sym}(2)$.

The superfield case is clear. The algorithm SMASH (Holt et al., 1996) shows that $\mathrm{GL}(2, 2) \circ \mathrm{GL}(4, 2)$ is semilinear, so we find no tensor product groups. Clearly there are no subfield groups, and no extraspecial-type groups.

In the classical case it is well-known that both $\mathrm{O}^+(8, 2)$ and $\mathrm{O}^-(8, 2)$ are subgroups of $\mathrm{Sp}(8, 2)$.

It is a trivial consequence of the results of Hiss and Malle (2001) and Lübeck (2001) that $\mathrm{GL}(8, 2)$ has no potentially maximal $\mathcal{C}_9$ subgroups. □

Proposition 5.2. *The group* $\mathrm{GL}(8, 2)$ *has* 238 *classes of irreducible subgroups, of which* 129 *are soluble and* 109 *are insoluble. The maximal irreducible subgroups of* $\mathrm{GL}(8, 2)$ *are those groups listed in Lemma* 5.1.

Proof. The list $\mathcal{M}_1$ consists of those groups listed in Lemma 5.1.

The function `HasComputableSubgroups()` returns `false` for both $\Gamma\mathrm{L}(4,4)$ and $\mathrm{Sp}(8,2)$, so we must consider both of these when constructing $\mathcal{M}_2$. We start with $\mathrm{Sp}(8,2)$ as this is the more straightforward task. The maximal subgroups of $\mathrm{PSp}(8,2) \cong \mathrm{Sp}(8,2)$ are listed in the $\mathbb{ATLAS}$. We add the groups $\mathrm{O}^-(8,2) : 2$, $\mathrm{O}^+(8,2) : 2$, $\mathrm{Sym}(10)$ and $\mathrm{PSL}(2,17)$ to $\mathcal{M}_2$. We do *not* append the superfield group $\mathrm{PSp}(4,4) : 2$ or the imprimitive group $\mathrm{Sp}(4,2) \wr 2$, as these will clearly not be maximal. We examine $\mathcal{M}_2$, and note that `HasComputableSubgroups()` returns `false` for the two orthogonal groups, both of which are in the $\mathbb{ATLAS}$. We therefore append $\mathrm{O}^\pm(8,2)$, as well as the groups $\mathrm{PSL}(2,7) : 2 \le \mathrm{O}^-(8,2) : 2$, $\mathrm{Sym}(9) \le \mathrm{O}^+(8,2) : 2$ and $\mathrm{Alt}(9) \le \mathrm{O}^+(8,2)$.

Next we consider $\Gamma\mathrm{L}(4,4)$. We include all maximal irreducible subgroups of $\Gamma\mathrm{L}(4,4)$ in $\mathcal{M}_2$, without concerning ourselves with the issue of containment in other groups. Our strategy relies on brute computational force rather than elegance: we start by applying Aschbacher's theorem to $\mathrm{GL}(4,4)$. Each of the potentially maximal subgroups of $\mathrm{GL}(4,4)$ is small enough to be dealt with automatically, as are the maximal reducible subgroups. We recursively find maximal subgroups of each group until we have a list $\mathcal{L}$ containing all self-normalizing subgroups of $\mathrm{SL}(4,4)$. Note that the recursion terminates as soon as we reach a nilpotent group L, as no proper subgroup of a nilpotent group is self-normalizing. We consider each group as a subgroup of $\Gamma\mathrm{L}(4,4)$, and $S := \Gamma\mathrm{L}(4,4)/\mathrm{SL}(4,4)$. We finish our construction of $\mathcal{M}_2$ with the following (automated) procedure:
For $T \le S$ and for each $G \in \mathcal{L}$ do

1. Let $\overline{T}$ be the preimage of T in $\Gamma\mathrm{L}(4,4)$.
2. Let $D := N_{\overline{T}}(G)$.
3. If D is irreducible and not equal to $\Gamma\mathrm{L}(4,4)$, add it to $\mathcal{M}_2$.

Note that this produces all groups G such that $\mathrm{SL}(4,4) \le G < \Gamma\mathrm{L}(4,4)$.

With $\mathcal{M}_1$ and $\mathcal{M}_2$ defined as in the preceding paragraphs, the output from `IrreducibleSubgroups(`$\mathcal{M}_1$, $\mathcal{M}_2$`)` contains the stated numbers of groups. A brief computation shows that each group lies in a unique Aschbacher class, and hence is maximal since there are no $\mathcal{C}_9$ groups. □

Lemma 5.3. *Let G be a maximal irreducible subgroup of* $\mathrm{GL}(9,2)$. *Then G lies in the following list:*

- $\mathcal{C}_2$. *Imprimitive groups:* $\mathrm{GL}(3,2) \wr \mathrm{Sym}(3)$.
- $\mathcal{C}_3$. *Superfield groups:* $\Gamma\mathrm{L}(3,8)$.
- $\mathcal{C}_7$. *Tensor induced groups:* $\mathrm{GL}(3,2) \wr \mathrm{Sym}(2)$.
- $\mathcal{C}_9$. $\mathrm{PSL}(3,4) \cdot \mathrm{Sym}(3)$.

Proof. The application of Aschbacher's theorem to find the potentially maximal irreducible geometric groups is completely straightforward, and will be omitted. Calculations similar to those described in Sections 3 and 4 show that $\mathrm{PSL}(3,4) \cdot \mathrm{Sym}(3)$ is the only potentially maximal subgroup of $\mathrm{GL}(9,2)$ in class $\mathcal{C}_9$. □

Proposition 5.4. *The group* $\mathrm{GL}(9, 2)$ *has* 36 *classes of irreducible subgroups, of which* 21 *are soluble and* 15 *are insoluble. The maximal irreducible subgroups of* $\mathrm{GL}(9, 2)$ *are those groups listed in Lemma* 5.3.

Proof. The list $\mathcal{M}_1$ consists of those groups listed in Lemma 5.3.

The only group in $\mathcal{M}_1$ for which `HasComputableSubgroups` returns `false` is $\Gamma\mathrm{L}(3, 8) = (7 \times \mathrm{PSL}(3, 8)) \cdot 3$. The simple subgroup $\mathrm{SL}(3, 8)$ is described in the $\mathbb{ATLAS}$. The only subgroups of $\Gamma\mathrm{L}(3, 8)$ which need to be included in $\mathcal{M}_2$ are the subfield group $(7 \times \mathrm{PSL}(3, 2)) \cdot 3$ and the groups G such that $\mathrm{SL}(3, 8) \leq G < \Gamma\mathrm{L}(3, 8)$.

A brief computation suffices to show that none of the listed groups are subgroups of one another, and the maximality claim follows. □

6. A new database of primitive groups

In this section we briefly describe the methods used to create our database of primitive groups, and discuss its structure.

We started by performing various checks on the list in Dixon and Mortimer (1988). In this paper, the primitive groups are listed by *cohort*, where two groups lie in the same cohort if and only if they have the same degree and their respective socles are permutation isomorphic. We used theoretical methods to check that the maximal groups in each cohort were as stated (there were a few errors), but we did *not* check that all possible cohorts were listed.

Given a cohort of primitive groups of degree d and with socle H, we used ad hoc techniques to construct a representation of $G := N_{\mathrm{Sym}(d)}(H)$. We then computed H and created a set $\mathcal{S}$ consisting of all subgroups of G/H. For each $S \in \mathcal{S}$ we looked at the preimage of S in G, and added it to the database if it was primitive. This method automatically verified that the minimal groups in each cohort were as stated, as well as guaranteeing that no primitive groups were omitted from the cohort.

Each entry of the database is labelled by a pair `(deg, num)` and consists of three fields `(Group, Name, ONS)`.

- `deg` is the degree of the group.
- `num` is the number of the group in the list of primitive groups of degree `deg`. We order the groups first by O'Nan–Scott type: Affine, Diagonal, Product, Almost Simple. Amongst the groups of affine type, the soluble groups come first. We then sort the families of groups by increasing order, and then by increasing rank. A consequence of this ordering is that the final two groups for each degree are Alt(`deg`) and Sym(`deg`).
- `Group` is the group itself. All groups are given as primitive permutation groups of degree `deg`, and most groups are 2-generated.
- If `Group` is insoluble then `Name` is the name of the group, in $\mathbb{ATLAS}$ notation. Otherwise, `Name` may be left blank. We have named all primitive insoluble groups of degree less than 1000, and many of the soluble groups as well. These names are only intended as a guide to the group's structure, and do not necessarily suffice to distinguish the groups from one another.

- `ONS` is the O'Nan–Scott type of `Group`.

The irreducible matrix group corresponding to the group G of affine type is returned by `MatrixQuotient(`G`)`.

7. Accuracy

It may be helpful to the reader to have a clear summary of the ways in which our work is reliant upon the work of others. For the overall structure of the primitive group classification, as well as for all primitive groups that are not of affine type, we have used Dixon and Mortimer's classification (Dixon and Mortimer, 1988). We have already discussed the extent to which we have assumed this to be correct. Clearly, we have made heavy use of Aschbacher's theorem; we have used the version given in his original paper (Aschbacher, 1984). We have used Kleidman and Liebeck's description of the structure of the geometric groups (Kleidman and Liebeck, 1990). In determining which geometric groups were potentially maximal we have used the algorithm SMASH (Holt et al., 1996), as implemented in MAGMA V2.9, for testing semilinearity and primitivity, and also the algorithm described in Leedham-Green and O'Brien (2002) to determine whether or not a matrix group is tensor induced.

In deciding if a group could have a $\mathcal{C}_9$ representation, our first port of call was the Modular Atlas (Jansen et al., 1995). If the group was not described there we used the work of Lübeck (2001) if it was a classical group in defining characteristic, and the work of Hiss and Malle (2001) otherwise. We did not re-check this work, however it should be noted that representations of quasisimple groups in dimension less than 10 have been well-understood for some time. When producing representations of such groups we made extensive use of MAGMA to check correctness.

Finally, in running our algorithm we are extremely reliant on both the maximal subgroups algorithms in Cannon and Holt (in press), the almost simple groups database in MAGMA, and our new conjugacy algorithms. Each group in the database with socle of order less than 1.6×10^7 either has its maximal subgroups listed in the $\mathbb{ATLAS}$, or has socle isomorphic to $\mathrm{PSL}(2, q)$ for $q \leq 317$. The maximal subgroups of almost simple groups with socle $\mathrm{PSL}(2, q)$ have been well-understood for over a century, see for instance Dickson (1901). This database has been extensively used by the general public for over 2 years, so we did not recheck its contents. Our conjugacy algorithm always produces a conjugating element if it claims that two subgroups are conjugate. In the rest of this section we will describe how we check that all of our resulting groups are distinct.

After completing our computations, we applied various checks to our results. The most complex test was to check that no two linear groups were conjugate under $\mathrm{GL}(n, p)$. The following lemma is well-known, see for instance Dixon and Mortimer, (1996, pp. 132–133).

Lemma 7.1. *Let $V := \mathbb{F}_p^{(n)}$. Let $G_1, G_2 \leq \mathrm{GL}(V)$ be irreducible, and let $H_1 := V : G_1$ and $H_2 := V : G_2$ be the corresponding primitive permutation groups of affine type. Then there exists a $g \in \mathrm{GL}(V)$ such that $G_1^g = G_2$ if and only if there exists an $s \in \mathrm{Sym}(p^n)$ such that $H_1^s = H_2$.*

By the above lemma, to check that our subgroups of $\mathrm{GL}(n, p)$ are distinct it suffices to test the conjugacy under the symmetric group of all groups in our new database of primitive groups of degree less than 1000 (which also includes the groups listed in Dixon and Mortimer (1988)).

The *signature* of a k-transitive permutation group G is a tuple consisting of $|G|$, k, the multiset of orbit lengths of the k-point stabilizer, the multiset of chief factors of G, and the orders of all groups in the derived series of G. For all integers d such that $1 \leq d \leq 999$ we partition the primitive groups of degree d into equivalence classes by signature. Those in classes of size 1 are ignored in later calculations.

The *extended signature* of a group G is the signature of G extended by an extra coordinate containing the multisets of isomorphism types of the abelian quotients in the derived series of G. We compute the derived subgroup G' of each of the remaining groups G. This is transitive, since G is primitive. We calculate the extended signature of each group G', and use this to partition the equivalence classes yet further. Once again, those groups that are in classes of size 1 can be ignored in further calculations.

We compute the signature of each Sylow subgroup of those G that remain, and partition the groups yet further. After this stage, only groups of affine type remain in classes of size greater than 1. We revert to the matrix group representation of the point stabilizer G_0 of each group G, and compute its conjugacy classes. For each conjugacy class of G_0 we then compute a triple consisting of the class size, the characteristic polynomial of a representative element and its minimal polynomial. After partitioning the groups with this test, very few remain in classes of size greater than 1, and we test that they are pairwise nonisomorphic. This is sufficient to show that for $1 \leq d \leq 999$ all groups of degree d in our database are not conjugate to one another in $\mathrm{Sym}(d)$.

We also checked our results by comparing with previous classifications. Our results agree with the corrected version of Short's classification (Short, 1991) of soluble groups. Our results for groups of degree less than 256 agree with both the integer sequence of numbers of primitive groups of degree d in the Encyclopedia of Integer Sequences (Sloane, 2002), as communicated by Hulpke in 2002, and the primitive groups database in GAP (GAP Group, 2002). Our list of insoluble subgroups of $\mathrm{GL}(8, 2)$ contains at least one copy of every group listed in Kondrat'ev (1987), but it is hard to compare results precisely, as an explicit list of the groups is not given. Our results for the insoluble subgroups of $\mathrm{GL}(9, 2)$ agree precisely with Kondrat'ev (1986). The numbers of soluble groups which we count in each case agree with the results of Eick and Höfling (in press).

8. Summary

There are too many matrix groups to be listed here: the groups themselves can be found at http://magma.maths.usyd.edu.au/users/colva, where they are listed by degree and then separated into soluble and insoluble.

In Table 2 we list the numbers of conjugacy classes of irreducible subgroups of $\mathrm{GL}(n, p)$ for $n > 1$ and $p^d < 1000$. We separate them into soluble and insoluble.

In Table 3 we list the number of primitive groups of diagonal type of degree d, and in Table 4 we list the number of primitive groups of product action type of degree d. In Table 5

Table 2
Number of soluble and insoluble irreducible subgroups of $GL(n, p)$

p	n	2	3	4	5	6	7	8	9
2	Soluble	2	2	10	2	40	2	129	21
	Insoluble	0	1	10	1	24	1	109	15
3	Soluble	7	9	108	16	324			
	Insoluble	0	2	37	18	147			
5	Soluble	19	22	509					
	Insoluble	3	11	138					
7	Soluble	29	62						
	Insoluble	4	14						
11	Soluble	42							
	Insoluble	6							
13	Soluble	62							
	Insoluble	6							
17	Soluble	75							
	Insoluble	5							
19	Soluble	77							
	Insoluble	9							
23	Soluble	54							
	Insoluble	4							
29	Soluble	100							
	Insoluble	10							
31	Soluble	114							
	Insoluble	12							

Table 3
The number N of primitive groups of degree d of diagonal type

d	60	168	360	504	660
N	5	5	16	4	5

Table 4
The number N of primitive groups of degree d of product action type

d	25	36	49	64	81	100	121	125	144	169	196	216
N	4	8	5	8	8	32	6	10	10	5	8	20
d	225	256	289	324	343	361	400	441	484	512	529	576
N	10	4	15	8	12	4	8	22	8	20	5	9
d	625	676	729	784	841	900	961					
N	49	28	28	36	4	8	6					

we list the number of primitive groups of degree d that are not symmetric or alternating. If there are no primitive groups of degree d other than $\mathrm{Alt}(d)$ and $\mathrm{Sym}(d)$ then d is omitted from the table. The number of almost simple primitive groups may be deduced from these tables.

Table 5
The number N of primitive groups of degree d other than Alt(d) and Sym(d)

d	5	6	7	8	9	10	11	12	13	14	15	16	17	18
N	3	2	5	5	9	7	6	4	7	2	4	20	8	2
d	19	20	21	22	23	24	25	26	27	28	29	30	31	32
N	6	2	7	2	5	3	26	5	13	12	6	2	10	5
d	33	35	36	37	38	40	41	42	43	44	45	47	48	49
N	2	4	20	9	2	6	8	2	8	2	7	4	2	38
d	50	52	53	54	55	56	57	59	60	61	62	63	64	65
N	7	1	6	2	6	7	3	4	7	12	2	6	72	11
d	66	67	68	71	72	73	74	77	78	79	80	81	82	83
N	5	8	5	8	2	14	2	2	4	8	2	153	8	4
d	84	85	89	90	91	97	98	100	101	102	103	104	105	107
N	4	4	8	2	8	12	2	36	9	3	8	2	9	4
d	108	109	110	112	113	114	117	119	120	121	122	125	126	127
N	2	12	2	8	10	2	3	2	21	55	5	43	17	13
d	128	129	130	131	132	133	135	136	137	138	139	140	144	149
N	5	2	5	8	2	1	3	12	8	2	8	2	15	6
d	150	151	152	153	155	156	157	158	162	163	164	165	167	168
N	2	12	2	4	1	7	12	2	5	10	2	5	4	7
d	169	170	171	173	174	175	176	179	180	181	182	183	186	190
N	73	5	4	6	2	4	4	4	2	18	2	2	1	4
d	191	192	193	194	196	197	198	199	200	203	208	210	211	212
N	8	2	14	2	8	9	2	12	2	1	3	4	16	2
d	216	220	223	224	225	227	228	229	230	231	233	234	239	240
N	20	3	8	2	10	4	2	12	2	4	8	4	8	2
d	241	242	243	244	248	251	252	253	255	256	257	258	263	264
N	20	2	34	4	1	8	2	7	2	242	13	2	4	2
d	266	269	270	271	272	273	275	276	277	278	280	281	282	283
N	1	6	2	16	2	6	2	6	12	2	22	16	2	8
d	284	285	286	289	290	293	294	297	300	307	308	311	312	313
N	2	1	2	95	5	6	2	2	9	13	2	8	2	16
d	314	315	317	318	324	325	330	331	332	336	337	338	341	343
N	2	3	6	2	8	12	4	16	2	7	20	2	2	88
d	344	347	348	349	350	351	353	354	357	359	360	361	362	364
N	6	4	2	12	2	9	12	2	5	4	18	90	5	9
d	367	368	369	373	374	378	379	380	381	383	384	389	390	396
N	8	2	3	12	2	9	16	2	2	4	2	6	2	2
d	397	398	400	401	402	406	409	410	416	419	420	421	422	425
N	18	2	14	15	2	4	16	2	5	8	2	24	2	1
d	431	432	433	434	435	439	440	441	443	444	449	450	455	456
N	8	2	20	2	4	8	2	22	8	2	14	2	2	2
d	457	458	461	462	463	464	465	467	468	479	480	484	487	488
N	16	2	12	6	16	2	5	4	2	4	2	8	12	2
d	491	492	495	496	499	500	503	504	506	509	510	511	512	513
N	12	2	8	10	8	2	4	6	1	6	2	1	56	12
d	520	521	522	523	524	525	527	528	529	530	540	541	542	547
N	7	16	2	12	2	6	2	7	63	5	8	24	2	16
d	548	553	557	558	560	561	563	564	567	569	570	571	572	576
N	2	1	6	2	4	2	4	2	5	8	2	16	2	9
d	577	578	585	587	588	593	594	595	599	600	601	602	607	608
N	21	2	4	4	2	10	2	2	8	2	24	2	8	2

Table 5 *(continued)*

d	613	614	616	617	618	619	620	625	626	630	631	632	641	642
N	18	2	2	16	2	8	3	696	8	2	24	2	16	2
d	643	644	647	648	651	653	654	657	659	660	661	662	666	671
N	8	2	8	2	5	6	2	2	8	7	24	2	4	2
d	672	673	674	676	677	678	680	683	684	691	692	693	701	702
N	6	24	2	28	9	2	2	8	2	16	2	4	18	2
d	703	709	710	715	719	720	727	728	729	730	733	734	739	740
N	4	12	2	2	4	2	12	2	499	13	12	2	12	2
d	741	743	744	750	751	752	756	757	758	759	761	762	765	769
N	2	8	2	1	16	2	5	26	2	1	16	2	2	18
d	770	773	774	775	780	781	784	787	788	792	797	798	806	809
N	2	6	2	1	2	1	36	8	2	2	6	2	8	8
d	810	811	812	816	819	820	821	822	823	824	827	828	829	830
N	2	20	2	2	6	22	12	2	8	2	8	2	18	2
d	839	840	841	842	853	854	857	858	859	860	861	863	864	871
N	4	6	114	5	12	2	8	2	16	2	4	4	2	1
d	877	878	880	881	882	883	884	887	888	891	900	903	907	908
N	12	2	1	20	2	18	2	4	2	4	8	4	8	2
d	910	911	912	919	920	929	930	937	938	941	942	945	946	947
N	3	16	2	16	2	12	2	24	2	12	2	2	4	8
d	948	953	954	960	961	962	967	968	969	971	972	977	978	980
N	2	16	2	8	132	5	16	2	2	8	2	10	2	2
d	983	984	990	991	992	993	997	998						
N	4	2	2	24	2	2	12	2						

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